Mathematics Behind the Options Calculator

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1 European Call

$$
\begin{aligned}
C &= Se^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2) \\
d_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

$$
\boxed{Se^{-qT}\,\phi(d_1) = Ke^{-rT}\,\phi(d_2)} \qquad \text{(Black-Scholes identity)}
$$

C Option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

Φ Standard normal CDF

ϕ Standard normal pdf

1.1 Delta

$$
\boxed{\Delta = \frac{\partial C}{\partial S} = e^{-qT}\Phi(d_1)}
$$

$$
\begin{aligned}
\Delta = \frac{\partial C}{\partial S} &= e^{-qT}\Phi(d_1) + Se^{-qT}\phi(d_1)\left(\frac{\partial d_1}{\partial S}\right) - Ke^{-rT}\phi(d_2)\left(\frac{\partial d_2}{\partial S}\right) \\
&= e^{-qT}\Phi(d_1) + \underbrace{Se^{-qT}\phi(d_1)\frac{1}{S\sigma\sqrt{T}} - Ke^{-rT}\phi(d_2)\frac{1}{S\sigma\sqrt{T}}}_{=0} \\
\Delta &= e^{-qT}\Phi(d_1)
\end{aligned}
$$

1.2 Gamma

$$\boxed{\Gamma = \frac{\partial^2 C}{\partial S^2} = e^{-qT}(\phi(d_1)\frac{1}{S\sigma\sqrt{T}})}$$

$$\begin{aligned}\Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} \\ &= e^{-qT}\frac{\partial}{\partial S}(\Phi(d_1)) \\ \Gamma &= e^{-qT}\phi(d_1)\frac{1}{S\sigma\sqrt{T}}\end{aligned}$$

1.3 Vega

$$\boxed{\mathcal{V} = \frac{\partial C}{\partial \sigma} = Se^{-qT}\phi(d_1)\sqrt{T}}$$

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = Se^{-qT}\phi(d_1)\left(\frac{\partial d_1}{\partial \sigma}\right) - Ke^{-rT}\phi(d_2)\left(\frac{\partial d_2}{\partial \sigma}\right)$$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}}\right) + \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}}\right) - \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{T}$$

$$\mathcal{V} = Se^{-qT}\phi(d_1)\sqrt{T}$$

1.4 Theta

$$\boxed{\Theta = \frac{\partial C}{\partial T} = -e^{-qT}\frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)}$$

$$\Theta = \frac{\partial C}{\partial T} = S\left(-qe^{-qT}\Phi(d_1) + e^{-qT}\phi(d_1)\frac{\partial d_1}{\partial T}\right) - K\left(-re^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial T}\right)$$

$$\frac{\partial d_1}{\partial T} = \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} := A$$

$$\frac{\partial d_2}{\partial T} = \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} - \frac{\sigma}{2\sqrt{T}} := A - \frac{\sigma}{2\sqrt{T}}$$

$$= Se^{-qT}(-q\Phi(d_1) + \phi(d_1)A) - Ke^{-rT}(-r\Phi(d_2) + \phi(d_2)(A - \tfrac{\sigma}{2\sqrt{T}}))$$

$$= -qSe^{-qT}\Phi(d_1) + rKe^{-rT}\Phi(d_2) + Ke^{-rT}\phi(d_2)\tfrac{\sigma}{2\sqrt{T}} + \underbrace{A(Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2))}_{=0}$$

$$= Ke^{-rT}\phi(d_2)\tfrac{\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(d_2) - qSe^{-qT}\Phi(d_1)$$

Using the Black-Scholes identity $Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)$, and expressing Θ as the negative:

$$\Theta = -e^{-qT}\frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)$$

1.5 Rho

$$\boxed{\rho = \frac{\partial C}{\partial r} = KTe^{-rT}\Phi(d_2)}$$

$$\rho = \frac{\partial C}{\partial r} = Se^{-qT}(\phi(d_1)\frac{\partial d_1}{\partial r}) - K(-Te^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial r})$$

$$\frac{\partial d_1}{\partial r} = \frac{T}{\sigma\sqrt{T}} = \frac{\sqrt{T}}{\sigma} = \frac{\partial d_2}{\partial r}$$

$$= Se^{-qT}\phi(d_1)\frac{\sqrt{T}}{\sigma} + KTe^{-rT}\Phi(d_2) - Ke^{-rT}\phi(d_2)\frac{\sqrt{T}}{\sigma}$$

$$= \frac{\sqrt{T}}{\sigma}\left(\underbrace{Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2)}_{=0}\right) + KTe^{-rT}\Phi(d_2)$$

$$\rho = KTe^{-rT}\Phi(d_2)$$

2 European Put

$$
\begin{aligned}
P &= Ke^{-rT}\,\Phi(-d_2)\ -\ Se^{-qT}\,\Phi(-d_1)\\
d_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

P Option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

Φ Standard normal CDF

ϕ Standard normal pdf

2.1 Delta

$$
\boxed{\Delta = \frac{\partial P}{\partial S} = -e^{-qT}\Phi(-d_1)}
$$

$$
\Delta = \frac{\partial P}{\partial S} = Ke^{-rT}\phi(d_2)\left(-\frac{\partial d_2}{\partial S}\right) - e^{-qT}\left(\Phi(-d_1) + S\phi(-d_1)\left(-\frac{\partial d_1}{\partial S}\right)\right)
$$

Using Black-Scholes identity:

$$
\Delta = -e^{-qT}\Phi(-d_1)
$$

2.2 Gamma

$$
\boxed{\Gamma = \frac{\partial^2 P}{\partial S^2} = e^{-qT}\,\frac{\phi(d_1)}{S\sigma\sqrt{T}} = Ke^{-rT}\,\frac{\phi(d_2)}{S^2\sigma\sqrt{T}}}
$$

Same as European Call derivation.

2.3 Vega

$$\boxed{\mathcal{V} = \frac{\partial P}{\partial \sigma} = Se^{-qT}\phi(d_1)\sqrt{T} = Ke^{-rT}\phi(d_2)\sqrt{T}}$$

$$\begin{aligned}
\mathcal{V} = \frac{\partial P}{\partial \sigma} &= Ke^{-rT}\left(\phi(d_2)\left(-\frac{\partial d_2}{\partial \sigma}\right)\right) - Se^{-qT}\left(\phi(d_1)\left(-\frac{\partial d_1}{\partial \sigma}\right)\right) \\
&= Se^{-qT}\phi(d_1)\left(\frac{\partial d_1}{\partial \sigma}\right) - Ke^{-rT}\phi(d_2)\left(\frac{\partial d_2}{\partial \sigma}\right) \\
\frac{\partial d_1}{\partial \sigma} &= -\frac{\ln\left(\frac{S}{K}\right) + (r-q)T}{\sigma^2\sqrt{T}} + \frac{1}{2}\sqrt{T} \\
\frac{\partial d_2}{\partial \sigma} &= -\frac{\ln\left(\frac{S}{K}\right) + (r-q)T}{\sigma^2\sqrt{T}} - \frac{1}{2}\sqrt{T} \\
\mathcal{V} &= Se^{-qT}\phi(d_1)\sqrt{T} = Ke^{-rT}\phi(d_2)\sqrt{T}
\end{aligned}$$

2.4 Theta

$$\boxed{\Theta = \frac{\partial P}{\partial T} = -e^{-qT}\frac{S\phi(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2) - qSe^{-qT}\Phi(-d_1)}$$

$$\begin{aligned}
\Theta = \frac{\partial P}{\partial T} &= K\left(-re^{-rT}\Phi(-d_2) + e^{-rT}\phi(d_2)\left(-\frac{\partial d_2}{\partial T}\right)\right) - S\left(-qe^{-qT}\Phi(-d_1) + e^{-qT}\phi(d_1)\left(-\frac{\partial d_1}{\partial T}\right)\right) \\
&= Ke^{-rT}\left(-r\Phi(-d_2) - \phi(d_2)\frac{\partial d_2}{\partial T}\right) - Se^{-qT}\left(-q\Phi(-d_1) - \phi(d_1)\frac{\partial d_1}{\partial T}\right) \\
\frac{\partial d_1}{\partial T} &= \frac{\left(\sigma^2 + 2r - 2q\right)T - 2\ln\left(\frac{S}{K}\right)}{4\sigma T^{3/2}} = A \\
\frac{\partial d_2}{\partial T} &= \frac{\left(\sigma^2 + 2r - 2q\right)T - 2\ln\left(\frac{S}{K}\right)}{4\sigma T^{3/2}} - \frac{\sigma}{2\sqrt{T}} = A - \frac{\sigma}{2\sqrt{T}} \\
&= Se^{-qT}\left(q\Phi(-d_1) + \phi(d_1)A\right) - Ke^{-rT}\left(r\Phi(-d_2) + \phi(d_2)\left(A - \frac{\sigma}{2\sqrt{T}}\right)\right) \\
&= qSe^{-qT}\Phi(-d_1) - rKe^{-rT}\Phi(-d_2) + Ke^{-rT}\phi(d_2)\frac{\sigma}{2\sqrt{T}} + A\left(Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2)\right) \\
&= qSe^{-qT}\Phi(-d_1) - Ke^{-rT}\left(r\Phi(-d_2) + \phi(d_2)\frac{\sigma}{2\sqrt{T}}\right) - Se^{-qT}\phi(d_1)\frac{\sigma}{2\sqrt{T}} \\
&= -Se^{-qT}\phi(d_1)\frac{\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2) - qSe^{-qT}\Phi(-d_1)
\end{aligned}$$

2.5 Rho

$$\boxed{\rho = \frac{\partial P}{\partial r} = -TKe^{-rT}\Phi(-d_2)}$$

$$\rho = \frac{\partial P}{\partial r} = K\left(-Te^{-rT}\Phi(-d_2) + e^{-rT}\phi(d_2)\left(-\frac{\partial d_2}{\partial r}\right)\right) - Se^{-qT}\phi(d_1)\left(-\frac{\partial d_1}{\partial r}\right)$$

$$\frac{\partial d_1}{\partial r} = \frac{T}{\sigma\sqrt{T}} = \frac{\sqrt{T}}{\sigma} = \frac{\partial d_2}{\partial r}$$

$$= -TKe^{-rT}\Phi(-d_2) \underbrace{- Ke^{-rT}\phi(d_2)\left(\frac{\sqrt{T}}{\sigma}\right) - Se^{-qT}\phi(d_1)\left(\frac{\sqrt{T}}{\sigma}\right)}_{=0}$$

$$\rho = -TKe^{-rT}\Phi(-d_2)$$

3 Binary Call (Cash-or-Nothing)

$$\begin{aligned} C_{\text{bin}} &= Q\, e^{-rT}\, \Phi(d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

C_{bin} Option price

Q Fixed cash payout if $S_T > K$

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

Φ Standard normal CDF

ϕ Standard normal pdf

3.1 Delta

$$\boxed{\Delta = \frac{\partial C_{\text{bin}}}{\partial S} = Qe^{-rT}\,\frac{\phi(d_2)}{S\sigma\sqrt{T}}}$$

$$\begin{aligned} \Delta = \frac{\partial C_{\text{bin}}}{\partial S} &= Qe^{-rT}\phi(d_2)\frac{\partial d_2}{\partial S} \\ &= Qe^{-rT}\phi(d_2)\frac{1}{S\sigma\sqrt{T}} \end{aligned}$$

3.2 Gamma

$$\boxed{\Gamma = \frac{\partial^2 C_{\text{bin}}}{\partial S^2} = -Qe^{-rT}\,\frac{d_1\phi(d_2)}{S^2\sigma^2 T}}$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = Qe^{-rT}\left(-\phi(d_2)\frac{1}{S^2\sigma\sqrt{T}} + \frac{1}{S\sigma\sqrt{T}}\cdot\frac{\partial\phi(d_2)}{\partial S}\right)$$

$$\frac{\partial\phi(d_2)}{\partial S} = \frac{\partial\phi(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial S} = -d_2\phi(d_2)\frac{1}{S\sigma\sqrt{T}}$$

$$= Qe^{-rT}\left(\frac{1}{S\sigma\sqrt{T}}\left(-d_2\phi(d_2)\frac{1}{S\sigma\sqrt{T}}\right) - \phi(d_2)\frac{1}{S^2\sigma\sqrt{T}}\right)$$

$$= -Qe^{-rT}\frac{1}{S\sigma\sqrt{T}}\left(d_2\phi(d_2)\frac{1}{S\sigma\sqrt{T}} + \phi(d_2)\frac{1}{S}\right)$$

$$= -Qe^{-rT}\frac{\phi(d_2)}{S\sigma\sqrt{T}}\left(\frac{d_2}{S\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{S\sigma\sqrt{T}}\right)$$

$$= -Qe^{-rT}\frac{\phi(d_2)}{S^2\sigma^2 T}\left(d_2 + \sigma\sqrt{T}\right)$$

$$\Gamma = -Qe^{-rT}\frac{d_1\phi(d_2)}{S^2\sigma^2 T}$$

3.3 Vega

$$\boxed{\mathcal{V} = \frac{\partial C_{\text{bin}}}{\partial\sigma} = -Qe^{-rT}\frac{d_1}{\sigma}\phi(d_2)}$$

$$\mathcal{V} = \frac{\partial C}{\partial\sigma} = Qe^{-rT}\phi(d_2)\cdot\frac{\partial d_2}{\partial\sigma}$$

$$= Qe^{-rT}\phi(d_2)\left(\frac{\partial d_1}{\partial\sigma} - \sqrt{T}\right)$$

$$\frac{\partial d_1}{\partial\sigma} = -\frac{\ln\left(\frac{S}{K}\right) + T(r-q)}{\sigma^2\sqrt{T}} + \tfrac{1}{2}\sqrt{T}$$

$$= Qe^{-rT}\phi(d_2)\left(-\frac{\ln\left(\frac{S}{K}\right) + T(r-q)}{\sigma^2\sqrt{T}} - \tfrac{1}{2}\sqrt{T}\right)$$

$$= -Qe^{-rT}\phi(d_2)\left(\frac{\ln\left(\frac{S}{K}\right) + T(r-q)}{\sigma^2\sqrt{T}} + \tfrac{1}{2}\sqrt{T}\right)$$

$$= -Qe^{-rT}\phi(d_2)\underbrace{\left(\frac{\ln\left(\frac{S}{K}\right) + T(r-q) + \tfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)}_{=\frac{d_1}{\sigma}}$$

$$= -Qe^{-rT}\phi(d_2)\frac{d_1}{\sigma}$$

3.4 Theta

$$\boxed{\Theta = \frac{\partial C_{\text{bin}}}{\partial T} = Qe^{-rT}\left[-r\Phi(d_2) - \frac{\phi(d_2)}{2T}\left(\frac{\ln(\frac{S}{K}) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right]}$$

$$\begin{aligned}
\Theta = \frac{\partial C}{\partial T} &= Q\left(-re^{-rT}\Phi(d_2) + \phi(d_2)\frac{\partial d_2}{\partial T}e^{-rT}\right)\\
&= Qe^{-rT}\left(-r\Phi(d_2) + \phi(d_2)\frac{\partial d_2}{\partial T}\right)\\
\frac{\partial d_2}{\partial T} &= \frac{\partial}{\partial T}\left(\frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)\\
&= -\frac{(\sigma^2 - 2r + 2q)T + 2\ln\left(\frac{S}{K}\right)}{4\sigma T^{3/2}}\\
&= Qe^{-rT}\left(-r\Phi(d_2) + \phi(d_2)\left(-\frac{2\ln\left(\frac{S}{K}\right) + (-2r + 2q + \sigma^2)T}{4\sigma T^{3/2}}\right)\right)\\
&= Qe^{-rT}\left(-r\Phi(d_2) + \frac{\phi(d_2)}{2T}\left(-\frac{\ln\left(\frac{S}{K}\right) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right)\\
&= Qe^{-rT}\left(-r\Phi(d_2) - \frac{\phi(d_2)d_2}{2T}\right)
\end{aligned}$$

3.5 Rho

$$\boxed{\rho = \frac{\partial C_{\text{bin}}}{\partial r} = Qe^{-rT}\left(-T\Phi(d_2) + \frac{\sqrt{T}}{\sigma}\phi(d_2)\right)}$$

$$\begin{aligned}
\rho = \frac{\partial C_{\text{bin}}}{\partial r} &= Q(-Te^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial r})\\
&= Qe^{-rT}(-T\Phi(d_2) + \phi(d_2)\frac{\sqrt{T}}{\sigma})\\
\rho &= Qe^{-rT}(-T\Phi(d_2) + \frac{\sqrt{T}}{\sigma}\phi(d_2))
\end{aligned}$$

4 Binary Put (Cash-or-Nothing)

$$\begin{aligned} P_{\text{bin}} &= Q\, e^{-rT}\, \Phi(-d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

P_{bin} Option price

Q Fixed cash payout if $S_T < K$

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

Φ Standard normal CDF

ϕ Standard normal pdf

4.1 Delta

$$\boxed{\Delta = \frac{\partial P_{\text{bin}}}{\partial S} = -Qe^{-rT}\frac{\phi(d_2)}{S\sigma\sqrt{T}}}$$

Same derivation as call but negative

4.2 Gamma

$$\boxed{\Gamma = \frac{\partial^2 P_{\text{bin}}}{\partial S^2} = Qe^{-rT}\frac{d_1\phi(d_2)}{S^2\sigma^2\, T}}$$

Same derivation as call

4.3 Vega

$$\boxed{\mathcal{V} = \frac{\partial P_{\text{bin}}}{\partial \sigma} = Qe^{-rT}\frac{d_1}{\sigma}\phi(d_2)}$$

Same derivation as call

4.4 Theta

$$\boxed{\Theta = \frac{\partial P_{\text{bin}}}{\partial T} = Qe^{-rT}\left[-r\Phi(-d_2) + \frac{\phi(d_2)}{2T}\left(\frac{\ln(\frac{S}{K}) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right]}$$

Nearly identical to call, just $\Phi(-d_2)$ instead of $\Phi(d_2)$ and adding additional term

4.5 Rho

$$\boxed{\rho = \frac{\partial P_{\text{bin}}}{\partial r} = Qe^{-rT}\left(-T\Phi(-d_2) - \frac{\sqrt{T}}{\sigma}\phi(d_2)\right)}$$

Nearly identical to call, just $\Phi(-d_2)$ instead of $\Phi(d_2)$ and subtracting additional term

5 American Call

$$C^A = \sup_{\tau \in \mathcal{T}[0,T]} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\,(S_\tau - K)^+\right]$$

(equals the European call when $q = 0$, since early exercise has no value).

C^A American call option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

τ Optimal stopping time $(0 \leq \tau \leq T)$

$\mathbb{E}^{\mathbb{Q}}$ Expectation under the risk-neutral measure

5.1 Delta

5.2 Gamma

5.3 Vega

5.4 Theta

5.5 Rho

6 American Put

$$P^A = \sup_{\tau \in \mathcal{T}[0,T]} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\left(K - S_\tau\right)^+\right]$$

P^A American put option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

τ Optimal stopping time ($0 \leq \tau \leq T$)

$\mathbb{E}^{\mathbb{Q}}$ Expectation under the risk-neutral measure

6.1 Delta

6.2 Gamma

6.3 Vega

6.4 Theta

6.5 Rho